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M-Order Generalized Frobenius Partitions with M Colors

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In 1984 G. E. Andrews introduced the idea of generalized Frobenius partitions (*Mem. Amer. Math. Soc.* **301**, 1984). Since that time they have been the subject of quite a bit of research. In this paper we present an alternative form of the generating function for M-order generalized Frobenius partitions with M colors and show that for $M=5$, 7, 11 they are closely related to ordinary partitions. We also present the Hardy–Ramanujan–Rademacher expansion for $\overline{c\phi}_M(n)$, the number of M-order generalized Frobenius partitions of n with M colors. © 1991 Academic Press, Inc.

Let $\overline{c\phi}_M(n)$ denote the number of generalized Fobenius partitions of n using M colors whose order under cyclic permutation of the colors is M. Clearly from this definition we have $\overline{c\phi}_M(n) \equiv 0 \pmod{M}$. It was somewhat of a surprise when it was discovered that in fact $\overline{c\phi}_M(n) \equiv 0 \pmod{M^2}$ [5]. In Section 1 of this paper we present an alternative form of the generating function for $\overline{c\phi}_M(n)$ which makes divisibility by this additional factor of M obvious.

In Section 2 we present a simple proof that $\overline{c\phi}_5(n) = 5p(5n-1)$, $\overline{c\phi}_7(n) = 7p(7n-2)$, and $\overline{c\phi}_{11}(n) = 11p(11n-5)$ based on some recent work of Garvan, Kim, and Stanton [4]. As a consequence of this result we are able to extend our congruences to higher powers of $M = 5$, 7, and 11 for certain values of n.

In Section 3 we present the Hardy–Ramanujan–Rademacher expansion for $\overline{c\phi}_M(n)$ using the generating function presented in Section 1.

Background information on generalized Frobenius partitions is in [1].

1. THE GENERATING FUNCTION FOR $\overline{c\phi}_M(n)$

In [7] it was shown that $\sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} \overline{c\phi}_M(n, m)\, q^n t^m$, where $\overline{c\phi}_M(n, m)$ is the number of generalized Frobenius partitions of n using M

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colors whose order under cyclic permutation of the colors is M and whose difference in colors between the top and bottom rows is m, is the coefficient of z^0 in

$$\sum_{d \mid M} \mu(d) \prod_{j=1}^{M/d} (z^d t^{dj} q^d; q^d)_\infty (z^{-d} t^{-dj}; q^d)_\infty .$$

We can rewrite this as

$$\sum_{d \mid M} \mu(d) \prod_{j=1}^{M/d} \frac{\sum_{m_j=-\infty}^{\infty} (-1)^{m_j} z^{dm_j} t^{djm_j} q^{\binom{m_j+1}{2} d}}{(q^d; q^d)_\infty}$$

using Jacobi's triple product identity.

In this same paper [7], it was shown that looking at the congruence class of m modulo M divides the partitions enumerated by $\overline{c\phi}_M(n)$ into M classes of equal size. The following theorem follows immediately.

THEOREM 1.

$$\sum_{n=1}^{\infty} \overline{c\phi}_M(n) q^n = \frac{M \sum q^{(1/2)\|\mathbf{m}\|^2}}{(q; q)_\infty^M},$$

where the sum on the right extends over all vectors $\mathbf{m}$ *in* $\mathbb{Z}^M$ *such that* $\mathbf{m} \cdot \mathbf{b} \equiv 1 \pmod{M}$ *with* $\mathbf{b} = (1, 2, ..., M)$ *and* $\mathbf{m} \cdot \mathbf{1} = 0$ *with* $\mathbf{1} = (1, 1, ..., 1)$.

Using the M vectors $\mathbf{v}_1 = (1, 0, ..., 0, -1)$ for $M \geqslant 2$,

$$\begin{aligned} \mathbf{v}_2 &= (-1, 1, ..., 0, 0) \\ &\vdots \\ \mathbf{v}_M &= (0, 0, ..., -1, 1) \end{aligned}$$

we can express each of the vectors m uniquely in the form $\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_M \mathbf{v}_M$ with each $\alpha_i \in \mathbb{Z}$ and $\alpha_1 + \alpha_2 + \cdots + \alpha_M = 1$. Thus Theorem 1 can be rewritten as

THEOREM 2. *For* $M \geqslant 2$

$$\sum_{n=1}^{\infty} \overline{c\phi}_M(n) q^n = \frac{M \sum q^{Q(\boldsymbol{\alpha})}}{(q; q)_\infty^M},$$

where the sum on the right extends over all vectors $\boldsymbol{\alpha}$ *in* $\mathbb{Z}^M$ *with* $\boldsymbol{\alpha} \cdot \mathbf{1} = 1$ *and* $Q(\boldsymbol{\alpha}) = (1/2) \sum_{i=1}^{M} (\alpha_i - \alpha_{i+1})^2$. (*Note* $\alpha_1 = \alpha_{M+1}$.)

By permuting the components of $\boldsymbol{\alpha}$ we easily see that $\sum q^{Q(\boldsymbol{\alpha})}$ is congruent to zero modulo M. Hence $\overline{c\phi}_M(n) \equiv 0 \pmod{M^2}$.

2. A CONNECTION WITH ORDINARY PARTITIONS FOR $M=5$, 7, 11

By Theorem 1 in [4] we have

$$\sum_{n=1}^{\infty} p(tn+r)\, q^n = \frac{\sum q^{Q(\boldsymbol{\alpha})}}{(q;q)_{\infty}^{t}}$$

for $(t, r)=(5, -1)$, $(7, -2)$, or $(11, -5)$ where $Q(\boldsymbol{\alpha})$ and the summation are the same as that given in Theorem 2 above. Hence we have

THEOREM 3. *For* $n \geqslant 1$,

$$\overline{c\phi}_5(n)=5p(5n-1), \qquad \overline{c\phi}_7(n)=7p(7n-2),$$

and

$$\overline{c\phi}_{11}(n)=11p(11n-5).$$

The first two parts of this theorem appeared in [6] and were proven by explicitly manipulating the generating functions for $\overline{c\phi}_5(n)$ and $\overline{c\phi}_7(n)$ to obtain Ramanujan's identities for the generating functions for $p(5n-1)$ and $p(7n-2)$.

Using the fact that

$$\begin{aligned} p(5^{\beta}n+\delta_{\beta}) &\equiv 0 \qquad (\mathrm{mod}\ 5^{\beta}) \\ p(7^{\beta}n+\gamma_{\beta}) &\equiv 0 \qquad (\mathrm{mod}\ 7^{[(\beta+2)/2]}) \\ p(11^{\beta}n+\lambda_{\beta}) &\equiv 0 \qquad (\mathrm{mod}\ 11^{\beta}), \end{aligned}$$

where δ_{β}, γ_{β}, and λ_{β} are the recipocals of 24 modulo 5^{β}, 7^{β}, and 11^{β}, respectively, [2], we have

THEOREM 4. *For* $\beta \geqslant 1$

$$\begin{aligned} \overline{c\phi}_5(\tfrac{1}{5}(5^{\beta}n+\delta_{\beta}+1)) &\equiv 0 \qquad (\mathrm{mod}\ 5^{\beta+1}) \\ \overline{c\phi}_7(\tfrac{1}{7}(7^{\beta}n+\gamma_{\beta}+2)) &\equiv 0 \qquad (\mathrm{mod}\ 7^{[(\beta+4)/2]}) \\ \overline{c\phi}_{11}(\tfrac{1}{11}(11^{\beta}n+\lambda_{\beta}+5)) &\equiv 0 \qquad (\mathrm{mod}\ 11^{\beta+1}). \end{aligned}$$

3. THE HARDY–RAMANUJAN–RADEMACHER EXPANSION FOR $\overline{c\phi}_M(n)$

Let $H(\boldsymbol{\alpha})$ be the quadratic form derived from $Q(\boldsymbol{\alpha})$ by replacing α_M with $1-\alpha_1-\cdots-\alpha_{M-1}$. $H(\boldsymbol{\alpha})$ takes the form $1+(1/2)\,\boldsymbol{\alpha}'T_M\boldsymbol{\alpha}-\mathbf{s}_M\boldsymbol{\alpha}$ for

$\boldsymbol{\alpha} \in \mathbb{Z}^{M-1}$, T is an $M-1 \times M-1$ even integral (diagonal entries are even), symmetric matrix, and $\mathbf{s}_M \in \mathbb{Z}^{M-1}$. For example, when $M=5$ we have

$$T_5 = \begin{pmatrix} 6 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \\ 3 & 1 & 4 & 2 \\ 4 & 3 & 2 & 6 \end{pmatrix}$$

and $\mathbf{s}_5 = (3, 2, 2, 3)$. T_M and $\mathbf{s}_M$ have similar forms for other values of M. In fact, we have the following results concerning T_M and $\mathbf{s}_M$ which we state without proof.

LEMMA 1. $|T_M| = M^3$.

LEMMA 2. *The diagonal entries of* T_M^{-1} *are* $(1/2M^2)\binom{M+1}{3}$.

LEMMA 3. $\mathbf{s}'_M T_M^{-1} \mathbf{s}_M = 2$.

The next two lemmas are essential in the development of the Hardy–Ramanujan–Rademacher expansion for $\overline{c\phi}_M(n)$. The first deals with the function $F(q) = \sum_{\boldsymbol{\alpha} \in \mathbb{Z}^{M-1}} q^{H(\boldsymbol{\alpha})}$ and the second is the well known transformation formula for the ordinary partition function, $P(q) = (q; q)_\infty^{-1}$ [2].

LEMMA 4.

$$F(e^{(2\pi i/k)(h+iz)}) = \frac{1}{M^{3/2}} z^{(1-M)/2} \sum_{m=0}^{\infty} S_{hk}(m)\, e^{-2\pi m/kN_M z},$$

where

$$S_{hk}(m) = k^{(1-M)/2} \sum_{\substack{H^*(\mathbf{c}) = m \\ \mathbf{c} \in \mathbb{Z}^{M-1}}} \sum_{\mathbf{a} \in \mathbb{Z}_k^{M-1}} e^{(2\pi i/k)(hH(\mathbf{a}) + (\mathbf{s}' T_M^{-1} - \mathbf{a})\mathbf{c})},$$

$H^*(\mathbf{c}) = (1/2)\,\mathbf{c}' N_M T_M^{-1} \mathbf{c}$, and N_M is the least positive integer such that $N_M T_M^{-1}$ is even-integral.

This lemma follows immediately by rewriting $\boldsymbol{\alpha}$ as $k\mathbf{c} + \mathbf{a}$ for $\mathbf{a} \in \mathbb{Z}_k^{M-1}$ and $\mathbf{c} \in \mathbb{Z}^{M-1}$ and then applying the transformation formula to the theta function $\theta(iz(T_M \mathbf{a} - \mathbf{s}_M), \pi z k T_M)$. The notation used here is the same as that in [3].

LEMMA 5.

$$P(e^{(2\pi i/k)(h+iz)}) = \omega_{h,k} z^{1/2} e^{\pi(z^{-1} - z)/12k} P(e^{(2\pi i/k)(h' + iz^{-1})}),$$

where $hh' \equiv 1 \pmod k$ *and* $\omega_{h,k}$ *is* 24*kth root of unity.*

From Cauchy's integral formula we have

$$\overline{c\phi}_M(n)=\frac{M}{2\pi i}\int_{\mathscr{C}}\frac{F(q)[P(q)]^M}{q^{n+1}}\,dq,$$

where $\mathscr{C}$ is any contour in the unit circle that winds around the origin once.

Using the method of Ford circles in [8] and letting $q=e^{(2\pi i/k)(h+i(z/k))}$ we obtain

$$\begin{aligned}\overline{c\phi}_M(n)&=M\sum_{\substack{0\leqslant h<k\leqslant N\\(h,k)=1}}\frac{i}{k^2}e^{-2\pi in(h/k)}\\&\quad\times\int_{z'_{hk}}^{z''_{hk}}F(e^{(2\pi i/k)(h+i(z/k))})[P(e^{(2\pi i/k)(h+i(z/k))})]^M e^{2\pi nz/k^2}\,dz\\&=\frac{1}{M^{1/2}}\sum_{\substack{0\leqslant h<k\leqslant N\\(h,k)=1}}\frac{i}{k^{5/2}}e^{-2\pi in(h/k)}\omega_{h,k}^M\\&\quad\times\int_{z'_{hk}}^{z''_{hk}}z^{1/2}e^{2\pi(n-M/24)z/k^2}\sum_{m_1,m_2=0}^{\infty}p_M(m_1)\,S_{hk}(m_2)\\&\quad\times e^{2\pi ih'm_1/k}e^{-\pi(2m_1+2m_2/N_M-M/12)/z}\,dz,\end{aligned}$$

where $p_M(m)$ is the coefficient of q^m in the expansion of $(q;q)_\infty^{-M}$.

It is not difficult to see that when $2m_1+2m_2/N_M-M/12$ is nonnegative and on the arcs of the circle K: $|z-1/2|=1/2$ from 0 to z'_{hk} and z''_{hk} to 0 that the above sum is $O(e^{4\pi|n|/N^2}N^{-1/2})$ and hence approaches zero as N tends to infinity. For more details see [3, 8, 9]

Thus $\overline{c\phi}_M(n)=1/\sqrt{M}\sum_{k=1}^{\infty}i/k^{5/2}\sum A_k(m_1,m_2)\int_{K^{(-)}}z^{1/2}e^{2\pi(n-M/24)z/k^2}e^{-\pi(2m_1+(2m_2/N_M)-(M/12))/z}\,dz$, where the innermost sum extends over all values of m_1 and m_2 such that $2m_1+2m_2/N_M-M/12$ is negative and $A_k(m_1,m_2)=\sum_{0\leqslant h<k\leqslant N,\,(h,k)=1}e^{-2\pi in(h/k)}e^{2\pi ih'm_1/k}\omega_{h,k}^M p_M(m_1)\,S_{hk}(m_2)$.

Using the fact that $\int_{K^{(-)}}z^{1/2}e^{uz+v/z}\,dz=-2\pi iv^{3/2}L_{3/2}(uv)$ and

$$L_{3/2}(w)=\begin{cases}\dfrac{d}{dw}\left(\dfrac{\sinh\sqrt{4w}}{\sqrt{\pi w}}\right) & \text{if } w>0\\[2ex] \dfrac{4}{3\sqrt{\pi}} & \text{if } w=0\\[2ex] \dfrac{d}{dw}\left(\dfrac{\sin\sqrt{-4w}}{\sqrt{-\pi w}}\right) & \text{if } w<0\end{cases}$$

we have

THEOREM 5. *For $M \geqslant 2$, $\overline{c\phi_M(n)} =$*

$$\frac{1}{\pi\sqrt{2M}} \sum_{k=1}^{\infty} \sqrt{k} \sum A_k(m_1, m_2)$$

$$\cdot \begin{cases} \dfrac{d}{dn}\left(\dfrac{\sinh(\pi/k)\sqrt{(2M/3-16m_1-16m_2/N_M)(n-M/24))}}{\sqrt{n-M/24}}\right) & \text{if } n-\dfrac{M}{24}>0 \\ \dfrac{4\pi^3}{3k^3}\left(\dfrac{M}{6}-4m_1-\dfrac{4m_2}{N_M}\right) & \text{if } n-\dfrac{M}{24}=0 \\ \dfrac{d}{dn}\left(\dfrac{\sin((\pi/k)\sqrt{(2M/3-16m_1-16m_2/N_M)(M/24-n)})}{\sqrt{M/24-n}}\right) & \text{if } n-\dfrac{M}{24}<0, \end{cases}$$

where the innermost sum extends over all values of m_1 and m_2 such that $2m_1 + 2m_2/N_M - M/12$ is negative and $A_k(m_1, m_2)$ is defined above.

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